Контрольная работа по теории вероятностей № 2.
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$\boxed{№4.}$

$$p(x|\alpha) = \alpha^k x^{k-1} e^{-\alpha x} \frac{1}{\Gamma(x)}, \ x \geq 0$$

$$\mathcal{L}(\alpha|x) = \prod_{i=1}^{n} \alpha^k x_i^{k-1} e^{-\alpha x_i} \frac{1}{\Gamma(x_i)} = \alpha^{kn} e^{-\alpha(\sum_{i=1}^n x_i)} \frac{1}{\prod_{i=1}^n \Gamma(x_i)}$$

$$L = \ln \mathcal{L} = kn \ln\alpha - \alpha \sum_{i=1}^n x_i - \ln \prod_{i=1}^n \Gamma(x_i)$$

$$\begin{cases} \frac{\partial L}{\partial \alpha} = \frac{kn}{\alpha} - \sum_{i=1}^n x_i \\ \frac{\partial L}{\partial \alpha} = 0 \end{cases}$$

$$\frac{kn}{\alpha} = \sum_{i=1}^n x_i$$

$$\alpha = \frac{kn}{\sum_{i=1}^n x_i}$$

$$\frac{\partial^2 L}{\partial \alpha^2} = -\frac{kn}{\alpha^2} < 0 \quad k > 0,$$

$$\Downarrow$$

$$\alpha_{ML} = \frac{kn}{\sum_{i=1}^n x_i} \qquad \text{Ответ: } \alpha_{ML} = \frac{kn}{\sum_{i=1}^n x_i}$$

$\boxed{№3.}$

$$\cancel{P_Y(y) = \frac{1}{6}}$$

$\boxed{№1.}$

$$\hat{C} = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$$

$$X \to X' = X + Z$$

$$Y \to Y' = Y + Z, \quad \operatorname{cov}(X', Y') = 0$$ (по сути диагонализовали матрицу)

$\sigma_{X'}^2 - ?$ $\sigma_{Y'}^2 - ?$ $\sigma_Z^2 - ?$

$\sigma_X^2 = 3$, $\sigma_Y^2 = 2$ – из матрицы $\hat{C}$

т. к. $Z \perp X$ $\mathbb{D}(X') = \mathbb{D}(X) + \mathbb{D}(Z)$

$$\hat{C}' = \begin{pmatrix} \sigma_{X'}^2 & 0 \\ 0 & \sigma_{Y'}^2 \end{pmatrix}$$

$$\operatorname{cov}(X', Y') = \mathbb{E}(X'Y') - \mathbb{E}(X')\mathbb{E}(Y') =$$

$$\left[\operatorname{cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = -1\right]$$

$$= \cancel{\mathbb{E}((X+Z)(Y+Z)) - \mathbb{E}(X+Z)\mathbb{E}(Y+Z)} = \operatorname{cov}(X+Z, Y+Z) =$$

$$\cancel{= \mathbb{E}(XY) + \mathbb{E}(XZ) + \mathbb{E}(YZ) + \mathbb{E}(Z^2)}$$

$$= \operatorname{cov}(X,Y) + \operatorname{cov}(X,Z) + \operatorname{cov}(Y,Z) + \operatorname{cov}(Z^2) =$$

$$= \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) + \mathbb{E}(XZ) - \mathbb{E}(X)\mathbb{E}(Z) + \mathbb{E}(YZ) -$$

$$- \mathbb{E}(\cancel{X})\mathbb{E}(Z) + \sigma_Z^2 = 0$$

$$-1 + \cancel{\mathbb{E}(X)\mathbb{E}(Z) - \mathbb{E}(X)\mathbb{E}(Z)} + \cancel{\mathbb{E}(Y)\mathbb{E}(Z) - \mathbb{E}(Y)\mathbb{E}(Z)} + \sigma_Z^2 = 0$$

[$\mathbb{E}(XZ) = \mathbb{E}(X)\mathbb{E}(Z)$ и так же с $Y$ потому что $Z \perp X, Y$.]

$\sigma_Z^2 = 1$

$\sigma_{X'}^2 = 1 + 3 = 4$ (тоже по независимости)

$\sigma_{Y'}^2 = 1 + 2 = 3$

Ответ: $\sigma_Z^2 = 1$, $\sigma_{X'}^2 = 4$, $\sigma_{Y'}^2 = 3$.

$\boxed{\text{№2.}}$

$\langle X^6 \rangle = \frac{5}{9}$, $\langle Y^4 \rangle = \langle X^2 Y^4 \rangle = 3$ $\qquad r(X,Y) - ?$

$\langle X^6 \rangle = 5!!\,\sigma_X^6 = 15\,\sigma_X^6 = \frac{5}{9}$

$\sigma_X^6 = \frac{\cancel{5}\,1}{9 \cdot \cancel{15}\,3} = \frac{1}{27}$

$\sigma_X^2 = \frac{1}{3}$

$\langle Y^4 \rangle = 3\,\sigma_Y^4 = 3 \Rightarrow \sigma_Y^4 = 1 \Rightarrow \sigma_Y^2 = 1$

$\langle X^2 Y^4 \rangle = \sigma_X \sigma_Y^3 \cdot 3 \cdot \mathbb{E}(XY) \Rightarrow \mathbb{E}(XY) = \frac{3}{3 \cdot \frac{1}{\sqrt{3}} \cdot 1} = \sqrt{3}$

$r(X,Y) = \frac{\mathbb{E}(XY)}{\sigma_X \sigma_Y} = \frac{\sqrt{3}}{\frac{1}{3} \cdot 1} = 3\sqrt{3}$

Ответ: $r(X,Y) = 3\sqrt{3}$.

$\boxed{\text{№3.}}$

$P(y) = \frac{1}{6}$, $y \in [0, 6]$

$F(y) = \frac{y}{6}$

$$\frac{4!}{(3-1)!\,(4-3)!} = \frac{24}{2} = 12$$

$$f_{Y_{(3)}}(y) = \frac{1}{108} y^2 (6 - y)$$