Вариант №47

Скрылошников Григорий

БФЗ 243.

**№1** $\hat{C} = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$ $\sigma_x^2 = 3$; $\sigma_y^2 = 2$.

$X' = X + Z$; $Y' = Y + Z$

$\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = -1.$

$X', Y'$ некоррелированные $\Rightarrow$ $\mathbb{E}[X'Y'] - \mathbb{E}[X']\mathbb{E}[Y'] = 0$

$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 = 3$

$\sigma_{x'}^2 = \langle x'^2 \rangle - \langle x' \rangle^2 = \sigma_x^2 + \sigma_z^2$

$\langle x'^2 \rangle = \langle (x+z)^2 \rangle = \langle x^2 \rangle + 2\langle x \rangle \langle z \rangle + \langle z^2 \rangle$

$\langle x' \rangle^2 = \langle x \rangle^2 + 2\langle x \rangle \langle z \rangle + \langle z \rangle^2$

$\sigma_{y'}^2 = \sigma_y^2 + \sigma_z^2$ — но аналогии

$\langle X'Y' \rangle = \langle XY \rangle + \langle XZ \rangle + \langle YZ \rangle + \langle Z^2 \rangle$

$\langle X' \rangle \langle Y' \rangle = \langle X \rangle \langle Y \rangle + \langle X \rangle \langle Z \rangle + \langle Y \rangle \langle Z \rangle + \langle Z \rangle^2$ $\Rightarrow$

$\Rightarrow \underbrace{\left(\langle XY \rangle - \langle X \rangle \langle Y \rangle\right)}_{-1} + \cancelto{0}{\left(\langle XZ \rangle - \langle X \rangle \langle Z \rangle\right)} + \cancelto{0}{\left(\langle YZ \rangle - \langle Y \rangle \langle Z \rangle\right)} +$

$+ \sigma_z^2 = 0.$

Получаем сис-му ур-й:

$\begin{cases} \sigma_z^2 = 1 \\ \sigma_{x'}^2 = 3 + \sigma_z^2 \\ \sigma_{y'}^2 = 2 + \sigma_z^2 \end{cases} \Rightarrow \boxed{\begin{cases} \sigma_z^2 = 1 \\ \sigma_{x'}^2 = 4 \\ \sigma_{y'}^2 = 3. \end{cases}}$

№2

$$\rho_x(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x} \exp\left(-(x-\mu^{0})^2/2\sigma_x^2\right)$$

$$\rho_y(y) = \frac{1}{\sqrt{2\pi}\,\sigma_y} \exp\left(-y^2/2\sigma_y^2\right)$$

Нормальное распределение с нулевым средним

$\langle x \rangle = \langle y \rangle = 0$

$\sigma_x^2 = \langle x^2 \rangle - \langle x \rangle^2 = \langle x^2 \rangle$

$\sigma_y^2 = \langle y^2 \rangle - \langle y \rangle^2 = \langle y^2 \rangle$

По формуле Вика.

$\langle x^6 \rangle = 15 \langle x^2 \rangle^3 = 15\,\sigma_x^6 = \frac{5}{3}$ ; $\sigma_x^6 = \frac{1}{27}$ ; $\sigma_x^2 = \frac{1}{3}$.

$\langle x^2 y^4 \rangle = 3 \langle x^2 \rangle \langle y^2 \rangle^2 + 12 \langle xy \rangle^2 \cdot \langle y^2 \rangle = 3\sigma_x^2 \cdot \sigma_y^4 + 12\sigma_y^2 \cdot \langle xy \rangle^2 = 3 \Rightarrow$

$\langle y^4 \rangle = 3 \langle y^2 \rangle^2 = 3\sigma_y^4 = 3$ ; $\sigma_y^4 = 1$ ; $\sigma_y^2 = 1$.

$\Rightarrow \underbrace{\sigma_x^2}_{=\frac{1}{3}} \cdot \underbrace{\sigma_y^4}_{=1} + 4\underbrace{\sigma_y^2}_{=1} \langle xy \rangle^2 = 1$

$4 \langle xy \rangle^2 = \frac{2}{3}$ ; $\langle xy \rangle^2 = \frac{1}{6}$ ; $\langle xy \rangle = \frac{1}{\sqrt{6}}$

$$r(x,y) = \frac{\operatorname{cov}(x,y)}{\sigma_x \sigma_y} = \frac{\frac{1}{\sqrt{6}}}{1 \cdot \frac{1}{\sqrt{3}}} = \frac{1}{\sqrt{2}}$$

$\operatorname{cov}(x,y) = \langle xy \rangle - \cancel{\langle x \rangle \langle y \rangle}^{0} = \langle xy \rangle$.

Ответ: $\frac{1}{\sqrt{2}}$.

№3 $Y_1, Y_2, Y_3, Y_4$ – выборка $\quad p_{Y_{(r)}}(y) = n C_{n-1}^{r-1} F^{r-1}(y) \cdot (1-F(y))^{n-r} p(y)$

$$p_{Y_i}(y) = \frac{1}{6} \text{ – равномерно}$$

n – количество

r – место (порядок) в выборке

$$p_{Y_{(3)}}(y) = 4 C_3^2 \cdot \left(\frac{1}{6}y\right)^2 \cdot \left(1 - \frac{1}{6}y\right) \cdot \frac{1}{6} =$$

$$= \cancel{12}^{3} \cdot \frac{1}{\cancel{36}} y^2 \cdot \cancel{\frac{5}{6}y}^{\frac{6-y}{6}} \cdot \frac{1}{6} = \cancel{\frac{5y^3}{108}}\ \frac{y^2(6-y)}{108} \quad y \in [0;6] \text{ – плотность вероятности}$$

$$P\left(1 < [Y_{(3)} - 3]^2 < 9\right) = P\left(1 < [Y_{(3)} - 3] < 3\right) =$$

$$= P\left(4 < Y_{(3)} < 6\right).$$

$$P(Y_{(3)}) = \int p_{Y_{(3)}} dy = \frac{1}{108}\left(2y^3 - \frac{y^4}{4}\right)$$

$$P\left(4 < Y_{(3)} < 6\right) = 1 - P\left(Y_{(3)} < 4\right) =$$

$$= 1 - \frac{1}{108} \cdot \left(2 \cdot 64 - \frac{\cancel{256}^{64}}{\cancel{4}}\right) = 1 - \frac{\cancel{64}^{\cancel{32}\ 16}}{\cancel{108}_{\cancel{54}\ 27}} = \frac{\cancel{8}\,11}{27} = \cancel{\frac{1}{3}}$$

$$P\left(1 < [Y_{(3)} - 3]^2 < 9\right) = \frac{11}{27}.$$

<u>№4</u> $X_1, X_2, \dots, X_n$ — выборка

$$p(x|\alpha) = \alpha^k x^{k-1} \frac{e^{-\alpha x}}{\Gamma(k)}, \quad x \geq 0$$

$$\ell(\alpha|\vec{x}) = \prod_{1\leq i\leq n} p(x_i|\alpha) = \alpha^{kn} \cdot \frac{1}{(\Gamma(k))^n} \cdot \prod_{1\leq i \leq n} x_i^{k-1} e^{-\alpha x_i} =$$

$$= \frac{\alpha^{kn}}{(\Gamma(k))^n} \cdot \exp\Big(-\alpha \cdot \sum_{1\leq i\leq n} x_i\Big) \cdot \Big(\prod_{1\leq i\leq n} x_i\Big)^{k-1}$$ — ф-я правдоподобия

$$L(\alpha|\vec{x}) = \ln \ell(\alpha|\vec{x}) = kn \ln\alpha - n \ln \Gamma(k) - \alpha \sum_{1\leq i\leq n} x_i +$$

$$+ (k-1) \ln\Big(\prod_{1\leq i\leq n} x_i\Big)$$

$$\frac{dL}{d\alpha} = \frac{kn}{\alpha} - \sum_{1\leq i\leq n} x_i = 0 \Rightarrow \alpha^* = \frac{kn}{\sum_{1\leq i\leq n} x_i}$$

$$\frac{d^2L}{d\alpha^2} = \frac{-kn}{\alpha^2} \qquad \frac{d^2L(\alpha^*)}{d\alpha^2} = \frac{-kn}{(kn)^2} \cdot \Big(\sum_{1\leq i\leq n} x_i\Big)^2 < 0$$ — максимум.

Ответ: $\dfrac{kn}{\sum_{1\leq i\leq n} x_i}$